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Lattice Fermions and the Chiral Anomaly

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Abstract

We show in the Wilson model that the contribution of the regular mass term to the four-divergence of the axial vector current in weak coupling perturbation theory is not zero in the chiral limit and is precisely the axial anomaly. Explicit breaking of chiral symmetry in the Wilson term is not relevant for the result. The ABJ anomaly is generated by the fermion mass term also with a chirally symmetric irrelevant term.

Introduction The proliferation of species known as *species doubling* is a major problem in formulating a lattice action for fermions. On a $3+1$ dimensional hypercubic euclidean lattice, there are in the spectrum 15 unwanted doublers in addition to the physical fermion. The doublers can be removed from the spectrum in the continuum limit (i.e., as the lattice spacing $a \to 0$), but only at a price. This, in essence, is the message of the no-go theorem of Nielsen and Ninomiya [1]. The price to be paid is to necessarily abandon one or more of the properties of (i) locality, (ii) hypercubic symmetry, (iii) reflection positivity, and (iv) chiral symmetry, in the lattice action for fermions [2].

The most popular model for lattice fermions, the Wilson model [3], solves the problem of doubling through an *irrelevant term*, the Wilson term, which breaks chiral symmetry explicitly. This specific choice, *viz.*, breaking chiral symmetry, turns out to have a more profound reason. It is generally accepted that in order to obtain the Adler-Bell-Jackiew (ABJ) anomaly in perturbation theory in the continuum limit, the irrelevant term in the lattice fermion action, needed to remove the spurious fermion doublers, should break chiral symmetry explicitly. Indeed, the contribution of the Wilson term to the four-divergence of the axial current is treated as the driving term for the ABJ anomaly [4,5]. This has been a major hurdle for lattice formulation of chiral gauge theories, because the regulator breaks the gauge symmetry explicitly.

We thus arrive at an impasse. Whereas, a lattice regulator preserving chiral symmetry is desirable for construction of a chiral gauge theory, it appears that precisely this symmetry needs to be broken explicitly in the irrelevant term of the lattice fermion action in order to produce the ABJ anomaly in the continuum limit. For a way out of this apparent impasse

it is instructive to look at the pioneering work of Lee and Nauenberg (LN) [6]. LN showed that helicity-flip interactions in QED, while forbidden for strictly massless electrons because of γ_5-invariance, survives in the chiral limit if one started with QED with massive electrons. The result of LN, therefore, suggests that the fermion mass term has more a dynamical role than just soft symmetry breaking. The clue was indeed picked up by Dolgov and Zakharov [7] who showed that the mass of a Dirac fermion could generate the ABJ anomaly in the axial current. The phenomenon is strongly reminiscent of ferromagnetism, which is triggered by a weak magnetic field but survives even after the latter is switched off.

In the following, we show that also on lattice the mass of a fermion generates the ABJ anomaly as in [6,7]. In the Wilson model, the continuum limit of the contribution of fermion mass m to the four-divergence of the axial current does not vanish in the chiral limit $m \to 0$, and, what is most interesting, coincides with the ABJ anomaly. The analysis needs rather mild constraints on the irrelevant term. Breaking chiral symmetry, for instance, is not required. Indeed, as we demonstrate in the following, the irrelevant term may just as well be chosen as chirally symmetric, and one can still generate the ABJ anomaly from the fermion mass term.

ABJ anomaly in Wilson model. In the following, we work in lattice QED with Wilson fermions. Contribution from the fermion mass to the four-divergence of the axial current in weak coupling perturbation in the second order is given by the amplitudes of diagrams (a), (b) and (c) in Fig.1.

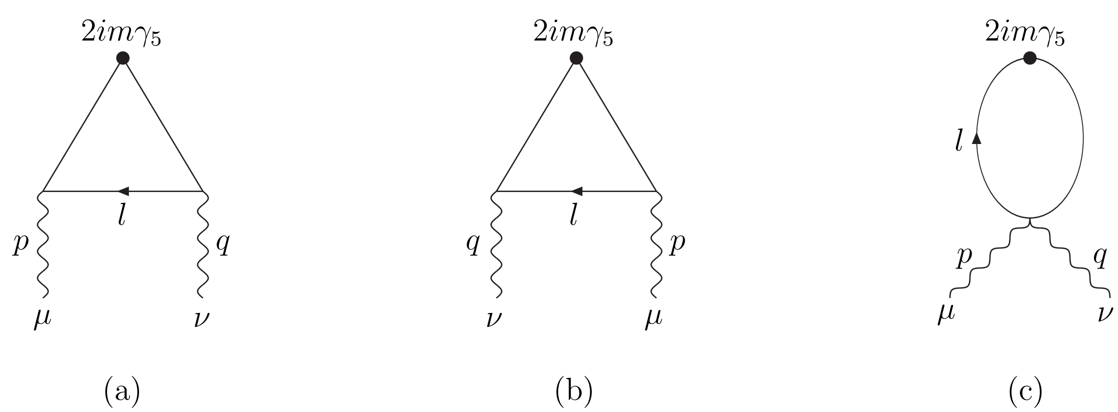

Fig.1

The external vertex in each diagram is $2im\gamma_5$. In the Wilson model, amplitude (c) vanishes, whereas, amplitudes (a) and (b) are equal and gauge invariant. The amplitudes (a) and (b) may, therefore, be combined and expressed in the notation used by Karsten and Smit [4] as

$$D^{(a+b)}_{\mu\nu} = -4g^2 m \int_l Tr[i\gamma_5 S(l-p) V_\mu(l-p,l) S(l) V_\nu(l,l+q) S(l+q)], \quad (1)$$

where the range of integration of the loop momentum l is the Brillouin zone ($-\frac{\pi}{a} \leq l_\alpha \leq \frac{\pi}{a}$). The fermion propagator $S(p)$ and the one-photon vertex $V_\mu(p,q)$ are

$$S(p) = \left[\sum_\mu \gamma_\mu \frac{\sin ap_\mu}{a} + \frac{M_c(ap)}{a} + m\right]^{-1}, \tag{2}$$

$$V_\mu = \gamma_\mu \cos\frac{a}{2}(p+q)_\mu + r\sin\frac{a}{2}(p+q)_\mu. \tag{3}$$

The Wilson term (M_c/a) which removes the doublers is given by

$$\frac{M_c(ap)}{a} = \frac{r}{a}\sum_\mu (1-\cos ap_\mu). \tag{4}$$

Gauge invariance dictates the tensor structure

$$D_{\mu\nu}^{(a+b)} \propto \epsilon_{\mu\nu\alpha\beta} p_\alpha q_\beta. \tag{5}$$

Reisz power counting [8,9], then gives for the lattice Feynman integral (1) an *effective* degree (see discussion below)

$$deg \;\; D_{\mu\nu}^{(a+b)} = -2. \tag{6}$$

The continuum limit of the lattice integral (1) is, therefore, given, according to the Reisz theorem [8,9], by the integral of continuum limit of the integrand

$$\begin{aligned}\lim_{a\to 0} D_{\mu\nu}^{(a+b)} &= ig^2 \epsilon_{\mu\nu\alpha\beta} p_\alpha q_\beta 16 \int_{-\infty}^{\infty} \frac{d^4 l}{(2\pi)^4} \frac{m^2}{(l^2+m^2)^3} \\ &= \frac{i}{2\pi^2} g^2 \epsilon_{\mu\nu\alpha\beta} p_\alpha q_\beta. \end{aligned} \tag{7}$$

It should be noted that the leading behaviour, according to the Reisz power counting, of the lattice Feynman integral (1) arises from the piece contributed by Dirac trace in the numerator which depends linearly on the Wilson term M_c/a. The degree of this term is 1. Two powers of the lattice spacing a, one each with p_α and q_β brings down the degree to -1. However, in the continuum limit the leading term vanishes. It is in this sense that the *effective* degree of (1) is -2 as stated in (6).

It is remarkable that the Wilson term (4) also generates a contribution equal in magnitude to (7) but opposite in sign. To see this, it is convenient to combine the usual mass term with the Wilson term

$$\frac{M(ap)}{a} = \frac{1}{a}\left[am + r\sum_\mu (1-\cos ap_\mu a)\right] \tag{8}$$

and consider the amplitude for the triangle diagrams (a) and (b) with this generalized momentum dependent mass at the vertex,

$$D_{\mu\nu}^M =$$

$$-2g^2 \int_l \frac{1}{a}\{M(al+aq)+M(al-ap)\} \;\; Tr\left[i\gamma_5 S(l-p)V_\mu(l-p,l)S(l)V_\nu(l,l+q)S(l+q)\right]. \tag{9}$$

The leading term of the gauge invariant form of the lattice Feynman integral thus obtained

$$D^M_{\mu\nu} = ig^2 \epsilon_{\mu\nu\alpha\beta} p_\alpha q_\beta \;\; 16 \;\; \int_l \frac{\left[M^2(l) - rM(l) \sum_\lambda (\frac{\sin^2 l_\lambda}{\cos l_\lambda})\right] \prod_\lambda \cos l_\lambda}{\left[M^2(l) + \sum_\lambda \sin^2 l_\lambda\right]^3} \tag{10}$$

has Reisz degree zero, and, therefore, is not amenable to the Reisz theorem. Thanks, however, to the trigonometric identity [4]

$$\begin{aligned} - rM(l) \sin^2 l_\lambda = \cos l_\lambda \left[\sin^2 l_\lambda - \frac{1}{4}(M^2(l) + \sum_\lambda \sin^2 l_\lambda)\right] \\ + \frac{1}{4}(M^2(l) + \sum_\lambda \sin^2 l_\lambda)^3 \frac{\partial}{\partial l_\lambda} \left[\frac{\sin l_\lambda}{(M^2(l) + \sum_\lambda \sin^2 l_\lambda)^2}\right], \end{aligned} \tag{11}$$

the lattice Feynman integral (10) is easily seen to vanish

$$D^M_{\mu\nu} = 0. \tag{12}$$

Combined with (7), the result (12) suggests that the *anomalies* arising from the 15 doublers cancel exactly the anomaly from the physical fermion [10]. In order that the result (7) translates into the familiar form

$$\langle \partial_\mu J_{\mu 5}(x) \rangle = \frac{i}{8\pi^2} g^2 F_{\mu\nu} \tilde{F}_{\mu\nu}. \tag{13}$$

in coordinate space, it is, therefore, necessary to regard the contribution $\langle X(x) \rangle$ from the Wilson term (4) as a piece of the lattice four-divergence of the axial vector current and write the axial Ward identity in the Wilson model as

$$\begin{aligned} \langle \frac{1}{2a} \{ \left[\overline{\psi}_x \gamma_\mu \gamma_5 U_{x,\mu} \psi_{x+\mu} + \overline{\psi}_{x+\mu} U^\dagger_{x,\mu} \gamma_\mu \gamma_5 \psi_x\right] - [x \to x - \mu] \} \rangle - \langle X(x) \rangle \\ = 2i \; m \; \langle \overline{\psi}_x \gamma_5 \psi_x \rangle, \end{aligned} \tag{14}$$

$$\langle X(x) \rangle = \frac{i\, r}{a} \langle 2\overline{\psi}_x \gamma_5 \psi_x - \{ \left[\overline{\psi}_x \gamma_5 U_{x,\mu} \psi_{x+\mu} + \overline{\psi}_{x+\mu} U^\dagger_{x,\mu} \gamma_5 \psi_x\right] + [x \to x - \mu] \} \rangle. \tag{15}$$

The contribution from the Wilson term thus plays a symmetric role with respect to the vector and the axial vector Ward identities. In either case, it is to be regarded as a piece of the four divergence of the respective currents on the lattice.

It is remarkable that the specific form of the irrelevant term, in the present case, the Wilson term (4), does not play any role in the derivation of the anomaly (7) except that it must remove all the doublers, thereby enabling the use of the Reisz theorem. Beyond this, we need gauge invariance to realize the tensor structure (5), and locality so that the differential Ward identities between the inverse propagator and photon vertices, e.g.,

$$V_\mu(p, p) = \frac{\partial}{\partial p_\mu} S^{-1}(p) \tag{16}$$

and its generalizations [4], hold.

Chiral anomaly in a chirally symmetric model. We consider a lattice fermion action with a chirally symmetric irrelevant term

$$\begin{aligned} S_F = & \sum_{x,\mu} \frac{1}{2a} \overline{\psi}_x \gamma_\mu [U_{x,\mu}\psi_{x+\mu} - U^\dagger_{x-\mu,\mu}\psi_{x-\mu}] + m \sum_x \overline{\psi}_x \psi_x \\ & + \sum_{x,\mu} \frac{1}{2a} \overline{\psi}_x \gamma_\mu \gamma_5 [2\psi_x - U_{x,\mu}\psi_{x+\mu} - U^\dagger_{x-\mu,\mu}\psi_{x-\mu}], \end{aligned} \tag{17}$$

proposed by us recently [11]. The free fermion propagator is now given by

$$\tilde{S}(p) = \left[\gamma_\mu \frac{\sin ap_\mu}{a} + i\gamma_\mu\gamma_5 \frac{1-\cos ap_\mu}{a} + m \right]^{-1}. \tag{18}$$

The irrelevant term

$$i\gamma_\mu\gamma_5 \frac{1-\cos ap_\mu}{a}, \tag{19}$$

as in other local chirally symmetric models [12], breaks hypercubic and reflection symmetries. The present model (17), however, is hermitian. This is an advantage in discussing reality properties in the continuum limit [13], and definitely in numerical simulations. Apart from the $\gamma_\mu\gamma_5$, in the irrelevant term in (17) one can recognize the scalar Wilson term. To recover hypercubic symmetry in the correlation functions, it seems necessary to average them over signs of the irrelevant term for each direction [14]. It is, however, possible that for gauge-invariant physical amplitudes the continuum limit itself takes care of the symmetry restoration as in Kogut-Susskind fermions.

The irrelevant term removes all the doublers from the spectrum [11]. This is evident from the absolute square of the inverse of the fermion propagator (18),

$$\left[\tilde{S}(p)\tilde{S}(p)^\dagger\right]^{-1} = \sum_\mu \left[\frac{\sin^2 ap_\mu}{a^2} + \frac{(1-\cos ap_\mu)^2}{a^2} \right] + m^2 - \frac{2}{a^2} \sum_{\mu\nu} \sigma_{\mu\nu}\gamma_5 \sin ap_\mu (1-\cos ap_\nu), \tag{20}$$

whose *trace* vanishes in the chiral limit $m = 0$ if and only if $p_\mu = 0$. The 1-loop vacuum polarization in QED has also been calculated and shows expected behaviour in the continuum limit. The calculation of the vacuum polarization will be reported elsewhere.

The fermion action (17) leads to the axial Ward identity

$$\langle \frac{1}{a} \sum_\mu (\tilde{J}_{\mu 5}(x) - \tilde{J}_{\mu 5}(x-\mu)) \rangle = 2im \langle \overline{\psi}_x \gamma_5 \psi_x \rangle, \tag{21}$$

with the axial current given by,

$$\tilde{J}_{\mu 5}(x) = \frac{1}{2} \left[\overline{\psi}_x \gamma_\mu \gamma_5 (1-\gamma_5) U_{x,\mu} \psi_{x+\mu} + \overline{\psi}_{x+\mu} U^\dagger_{x,\mu} (1-\gamma_5) \gamma_\mu \gamma_5 \psi_x \right]. \tag{22}$$

The diagrams in Fig.1 represent, as in the Wilson case, the emission of two photons arising from the right hand side of the Ward identity (21) (the lower order diagrams can be shown to vanish). To construct the lattice amplitudes we need one- and two-photon vertices

$$
\begin{aligned}
\tilde{V}_\mu(p,q) &= \gamma_\mu \cos\frac{a}{2}(p+q)_\mu + i\gamma_\mu\gamma_5 \sin\frac{a}{2}(p+q)_\mu, \\
\tilde{V}_{\mu\nu}(p,q) &= \delta_{\mu\nu} a \left[-\gamma_\mu \sin\frac{a}{2}(p+q)_\mu + i\gamma_\mu\gamma_5 \cos\frac{a}{2}(p+q)_\mu\right].
\end{aligned} \tag{23}
$$

The amplitudes of the diagrams (a) and (b), individually are not gauge invariant. One finds instead

$$
\begin{aligned}
\sum_\mu \frac{2}{a}\sin\frac{a}{2}p_\mu \tilde{D}^{(a+b)}_{\mu\nu} &= 4g^2 m \int_l Tr[i\gamma_5 \tilde{S}(l)\{\tilde{V}_\nu(l,l+q) - \tilde{V}_\nu(l+p,l+p+q)\}\tilde{S}(l+p+q)] \\
&= -8g^2 m \sum_\mu \int_l Tr[i\gamma_5 \tilde{S}(l)\tilde{V}_{\mu\nu}(l,l+p+q)\tilde{S}(l+p+q)] \sin\frac{a}{2}p_\mu \\
&= -\sum_\mu \frac{2}{a}\sin\frac{a}{2}p_\mu \tilde{D}^{(c)}_{\mu\nu}.
\end{aligned} \tag{24}
$$

Thus, in the present case only the sum of amplitudes of all the three diagrams (a), (b) and (c) is gauge-invariant and has the structure

$$
\tilde{D}^{(a+b+c)}_{\mu\nu} \propto \epsilon_{\mu\nu\alpha\beta} p_\alpha q_\beta. \tag{25}
$$

The amplitude for the diagram (c) is a function of $(p+q)$, and thus symmetric in p and q. Therefore, it cannot contribute to the gauge invariant structure (25). The leading terms in the amplitudes for the diagrams (a) and (b) are of degree 1 by Reisz power counting, as before, but in the present case they do not contribute because of vanishing trace of odd number of γ-matrices. The coefficient of $p_\alpha q_\beta$ in the gauge-invariant structure (25) has, therefore, an effective degree -2 as in the case of Wilson model. Reisz theorem allows, as before, to take the continuum limit $a \to 0$ within the lattice integral. Due to this, the dependence of the vertices $\tilde{V}_\mu(l-p,l)$ and $\tilde{V}_\nu(l,l+q)$ in

$$
\tilde{D}^{(a)}_{\mu\nu} = -2g^2 m \int_l Tr\left[i\gamma_5 \tilde{S}(l-p)\tilde{V}_\mu(l-p,l)\tilde{S}(l)\tilde{V}_\nu(l,l+q)\tilde{S}(l+q)\right] \tag{26}
$$

on the external momenta p and q respectively can be ignored. This can be easily verified by taking the derivative of $\tilde{V}_\mu$ with respect to p_μ. The resulting contribution vanishes in the continuum limit. In extracting the gauge invariant structure (25), the dependence of only the propagators $\tilde{S}(l-p)$ and $\tilde{S}(l+q)$ on external momenta is relevant.

The gauge invariant contributions from the diagrams (a) and (b) thus coincide with the same in the Wilson model and ABJ anomaly (7) is reproduced in the present model (17),

$$
\lim_{a\to 0} \tilde{D}^{(a+b)}_{\mu} = \frac{i}{2\pi^2} g^2 \epsilon_{\mu\nu\alpha\beta} p_\alpha q_\beta. \tag{27}
$$

Conclusions. At finite lattice spacing a, there are no anomalous Ward identities on the lattice. Anomalies, if any, appear only in the continuum limit through correspondences assumed between lattice operators and their continuum counterparts [10]. It is natural that the contribution from the irrelevant term, a lattice artifact, is identified as the generator of the anomaly in axial Ward identity [4,5].This identification, it should be emphasized, is at best a plausible assumption. Indeed, in the case of lattice Ward identity for the vector

current, the contribution from the irrelevant term is included in the definition of the four-divergence of the vector current in the continuum [4]. In the present approach, however, we have identified the contribution of the physical mass of the fermion as the generator of the ABJ anomaly. The ABJ anomaly in this approach consists in the difference in the continuum limit of the four-divergence of the axial current in a gauge theory with massless ($m = 0$) fermion and that obtained in the chiral limit $m \to 0$ starting with a massive fermion. This is how the results of Lee and Nauenberg [6] and of Dolgov and Zakharov [7] in continuum QED are realized on lattice.

REFERENCES

[1] H.B. Nielsen and M. Ninomiya, Nucl. Phys. B185 (1981) 541.

[2] A. Pelissetto, Ann. Phys. (N.Y.) 182 (1988) 177; M. Pernicci, Phys. Lett. B346 (1995) 99.

[3] K.G. Wilson, in *New Phenomena in Subnuclear Physics* (Erice, 1975), ed. A. Zichichi (Plenum Press, New York, 1977), p.69.

[4] L.H. Karsten and J. Smit, Nucl. Phys. B127 (1981), 103.

[5] W. Kerler, Phys. Rev. D 23 (1981) 2384; E. Seiler and I.O. Stamatescu, Phys. Rev. D 25 (1982) 2177; T. Hattori and H. Watanabe, hep-lat/9708007.

[6] T.D. Lee and M. Nauenberg, Phys. Rev. B 133 (1964) 1549.

[7] A.D. Dolgov and V.I. Zakharov, Nucl. Phys. B27 (1971) 525; K. Huang, in *Quarks, Leptons and Gauge Fields*(World Scientific, 1982), p219.

[8] T. Reisz, Comm. Math. Phys. 116 (1988) 81; Comm. Math. Phys. 116 (1988) 573.

[9] M. Luescher, in *Fields, strings and critical Phenomena* (Les Houches, 1988), eds. E. Brezin and J. Zinn-Justin, (North Holland, Amsterdam, 1989), p451.

[10] K. Fujikawa, Z. Phy. C25 (1984) 179.

[11] H. Banerjee and Asit K. De, Nucl. Phys. B (Proc. Suppl.) 53 (1997) 641.

[12] L. Jacobs, Phys. Rev. Lett. 51 (1983) 172; D. Weingarten and B. Velikson, Nucl. Phys. B270 (1986) 10; J.L. Alonso and J.L. Cortes, Phys. Lett. B187 (1987) 146; I.O. Stamatescu and T.T. Wu, Nucl. Phys. B (Proc. Suppl.) 42 (1995) 838.

[13] J. Smit, Nucl. Phys B (Proc. Suppl.) 4 (1988) 451.

[14] N. Sadooghi and H.J. Rothe, Phys. Rev. D55 (1997) 6749.